Math 102

Krishanu Sankar

October 2, 2018

Announcements

- Office Hours this week - W 11:30-1 and Th 12:30-2.

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- In-class
  - I'll post slides on Canvas before lecture, and include extra practice questions at the end of the slides.
  - If lecture is going too slowly and you want more practice, you can work on these questions.

Goals Today

- Linear Approximation and Newton's Method
  - Similarities and differences
  - What function and starting point to pick?
- Aphids and Ladybugs: a model
  - Using derivatives to deduce features of a function
  - Describing qualitative behavior
- Graph Sketching using derivatives

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General theme: more open-ended problem solving!

Warmup - Linear Approximation and Newton's Method

Question: Estimate the value of $\sqrt{15}$. Using linear approximation, the best choice will be:

1. $f(x) = \sqrt{x} \quad ; \quad x_0 = 16$
2. $f(x) = \sqrt{x} - 15 \quad ; \quad x_0 = 4$
3. $f(x) = \sqrt{x + 15} \quad ; \quad x_0 = 0$
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$$f(15) \approx f(16) + f'(16)(15 - 16)$$
$$= 4 + \frac{1}{2 \cdot 4}(-1) = \frac{31}{8}$$

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$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= 4 - \frac{16 - 15}{2 \cdot 4} = \frac{31}{8}$$

Ladybugs and Aphids

$x =$ size of aphid population.

$P(x) =$ predation rate $= \frac{30x^3}{20^3+x^3}$

$G(x) =$ growth rate $= 0.5x$

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Let $R(x)$ be the net rate of change.

$$
\begin{aligned}
R(x) &= G(x) - P(x) \\
&= 0.5x - \frac{30x^3}{20^3 + x^3}
\end{aligned}
$$

Question: Why might I care about finding which values of x makes $R(x) = 0$?

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Warning: x is the *aphid population*, **not** time!

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Question: What are the positive values of x such that $R(x) = 0$? Let's solve...

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$$\implies \frac{1}{2} = \frac{30x^2}{20^3 + x^3}$$

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$$0.5 - \frac{30x^2}{20^3 + x^3} = 0$$

$$\implies \frac{1}{2} = \frac{30x^2}{20^3 + x^3}$$

$$\implies 8000 + x^3 = 60x^2 \implies x^3 - 60x^2 + 8000 = 0$$

Sketching $f(x) = x^3 - 60x^2 + 8000$

$$x \approx 0 \implies f(x) \approx 8000$$
$$x \to \pm\infty \implies f(x) \to \pm\infty$$

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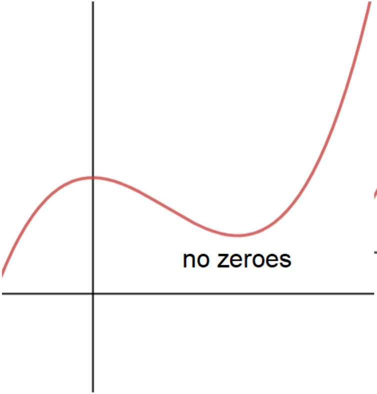
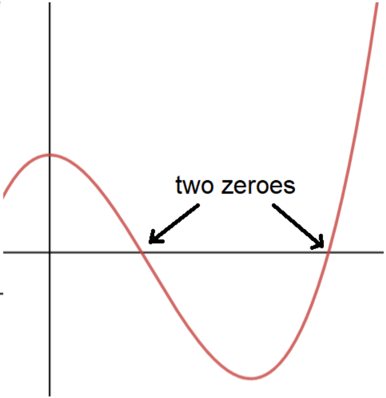
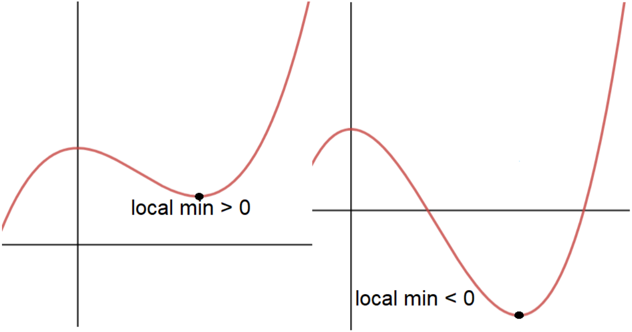
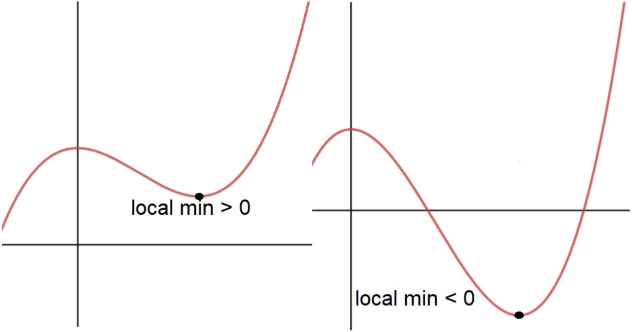
Question: Which sketch is more accurate?







Strategy: We can calculate the y-coordinate of the local minimum.



Strategy: We can calculate the y-coordinate of the local minimum.
We will do this by solving $f'(x) = 0$, then plugging that x-value into $f(x)$.

$$f(x) = x^3 - 60x^2 + 8000$$
$$\implies f'(x) = 3x^2 - 120x = 0$$
$$\implies 3x^2 = 120x \implies x = 0, 40$$

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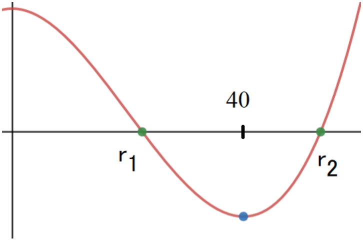
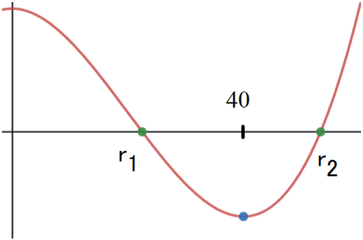
Plugging in,

$$f(40) = 64000 - 96000 + 8000 = -24000$$

which is negative. Therefore, $f(x)$ has two positive zeroes.



Newton's Method



Question: The local minimum occurs at $x = 40$. If we want to estimate the **larger root**, what would be a good x_0 to start with?

(A) $x_0 = 40$ (B) $x_0 = 20$

(C) $x_0 = 0$ (D) $x_0 = 50$

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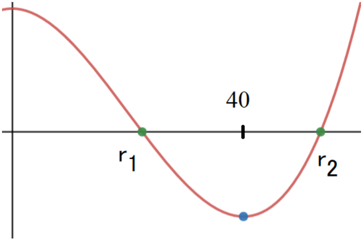

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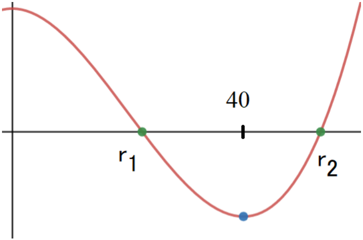

Question: The local minimum occurs at $x = 40$. If we want to estimate the **smaller root**, what would be a good x_0 to start with?

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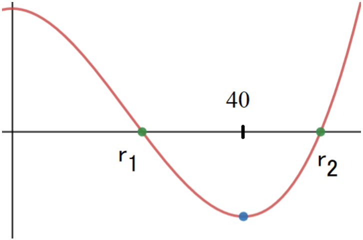

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Newton's Method

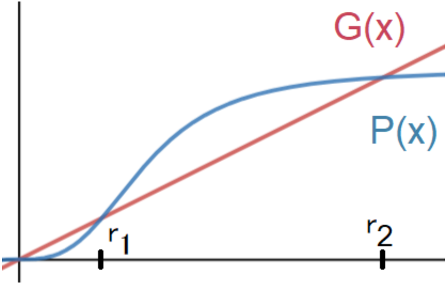
$$f(x) = x^3 - 60x^2 + 8000$$
$$f'(x) = 3x^2 - 120x$$

Iteratively use the formula $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$, and stop when satisfied. For example, if $x_0 = 20$,

$$\begin{aligned} x_1 &= 20 - \frac{f(20)}{f'(20)} \\ &= 20 - \frac{-8000}{-1200} \\ &= 20 - \frac{20}{3} \approx 13.33 \end{aligned}$$

Ladybugs and Aphids

Question: Suppose that $x > r_2$. What happens to the aphid population over time?



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2. It shrinks to zero.
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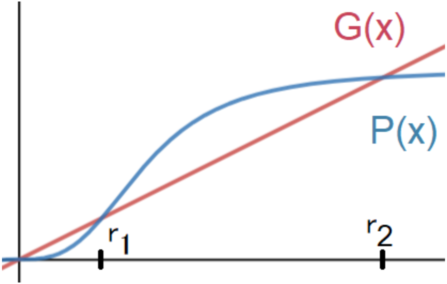

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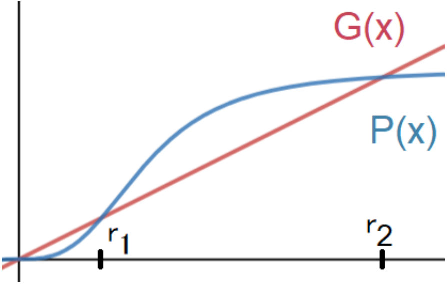

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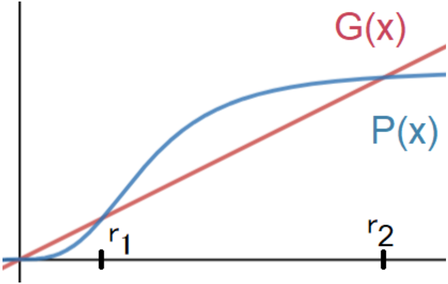

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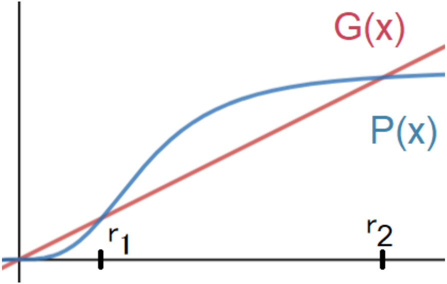

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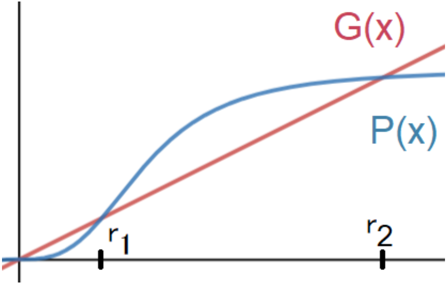

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Graph Sketching

- Previously, we learned to sketch the graph of a rational function using **asymptotic behavior**.

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- Previously, we learned to sketch the graph of a rational function using **asymptotic behavior**.
- This provides rough sketches. We'd like to use calculus to create more accurate drawings of functions. For example,
  - Are there zeroes?
  - Are there local maxima and minima?
  - Are there inflection points?
  - Can we use these features to make our drawing to scale?

Graph Sketching

Question: Create a to-scale graph of

$$f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - 3x^2 + 4$$

- Behavior at $x \approx 0$ and $x \to \infty$
- $f'(x)$: function is increasing vs. decreasing. Label critical points.
- $f''(x)$ function is concave up vs concave down. Label inflection points (when concavity changes sign).

Recap

- Linear Approximation and Newton's Method
  - Similarities and differences
  - What function and starting point to pick?
- Aphids and Ladybugs: a model
  - Using derivatives to deduce features of a function
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Additional Exercises - tangent lines

- Question: Find the equation of the line tangent to $y = 2x^3 - 9x^2 + 6$ which is parallel to $y = -12x + 1$.
- Question: Find the equation of the line tangent to $y = 4 - x^2$ which passes through $(0, 5)$.

We may do questions similar to these in a future class!